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Approximation Theorems for Double Orthogonal Series, II

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Let $\{\phi_{ik}(x): i, k = 1, 2, ...\}$ be an orthonormal system on a positive measure space and let $\{a_{ik}\}$ be a sequence of real numbers for which $\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 \lambda^2(i, k) < \infty$ for an appropriate nondecreasing (in i and in k) double sequence $\{\lambda(i, k)\}$ of positive numbers. Then the sum $f(x)$ of the double orthogonal series $\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik} \phi_{ik}(x)$ converges in the sense of L^2-metric as well as in the sense of a.e. pointwise convergence. We study the rate of a.e. approximation to $f(x)$ by the rectangular partial sums $s_{mn}(x) = \sum_{i=1}^{m} \sum_{k=1}^{n} a_{ik} \phi_{ik}(x)$, by the arithmetic means $\sigma_{mn}^{10}(x) = m^{-1} \sum_{i=1}^{m} s_{in}(x)$ with respect to m, and by the arithmetic means $\sigma_{mn}^{11}(x) = m^{-1} n^{-1} \sum_{i=1}^{m} \sum_{k=1}^{n} s_{ik}(x)$ with respect to m and n. Two special cases of our main results read as follows.

THEOREM 2′. *If* $(*)$ $\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 (i^{2\alpha} + k^{2\beta})([\log\log(i+3)]^2 + [\log\log(k+3)]^2) < \infty$ *for some* $0 < \alpha, \beta < 1$, *then* $\sigma_{mn}^{11}(x) - f(x) = o_x\{m^{-\alpha} + n^{-\beta}\}$ *a.e. as* $\min\{m, n\} \to \infty$.

THEOREM 3′. *If condition* $(*)$ *is satisfied for some* $0 < \alpha, \beta < 1/2$, *then* $\{m^{-1} n^{-1} \sum_{i=1}^{m} \sum_{k=1}^{n} [s_{ik}(x) - f(x)]^2\}^{1/2} = o_x\{m^{-\alpha} + n^{-\beta}\}$ *a.e. as* $\min\{m, n\} \to \infty$.

1. INTRODUCTION: PREVIOUS RESULTS

Let $\{\phi_{ik}(x): i, k = 1, 2, ...\}$ be an orthonormal system (abbreviated ONS) on a positive measure space $(X, \mathscr{F}, \mu)$. We will consider the double orthogonal series

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik} \phi_{ik}(x), \tag{1.1}$$

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where $\{a_{ik}: i, k=1, 2, ...\}$ is a double sequence of real numbers for which

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 < \infty. \tag{1.2}$$

By the Riesz–Fischer theorem there exists a function $f(x) \in L^2 = L^2(X, \mathscr{F}, \mu)$ such that series (1.1) is the Fourier series of $f(x)$ with respect to the system $\{\phi_{ik}(x)\}$. In particular, the rectangular partial sums

$$s_{mn}(x) = \sum_{i=1}^{m} \sum_{k=1}^{n} a_{ik} \phi_{ik}(x) \qquad (m, n=1, 2, ...)$$

converge to $f(x)$ in the L^2-metric:

$$\int [s_{mn}(x) - f(x)]^2 \, d\mu(x) \to 0 \qquad \text{as} \quad \min\{m, n\} \to \infty.$$

Here and in the sequel the integrals are taken over the entire space X.

Denote by $\sigma_{mn}(x)$ the arithmetic means of the rectangular partial sums with respect to m and n:

$$\sigma_{mn}(x) = \sigma_{mn}^{11}(x) = \frac{1}{mn} \sum_{i=1}^{m} \sum_{k=1}^{n} s_{ik}(x)$$
$$= \sum_{i=1}^{m} \sum_{k=1}^{n} \left(1 - \frac{i-1}{m}\right)\left(1 - \frac{k-1}{n}\right) a_{ik} \phi_{ik}(x),$$

and by $\sigma_{mn}^{10}(x)$ the arithmetic means with respect to only m:

$$\sigma_{mn}^{10}(x) = \frac{1}{m} \sum_{i=1}^{m} s_{in}(x) = \sum_{i=1}^{m} \sum_{k=1}^{n} \left(1 - \frac{i-1}{m}\right) a_{ik} \phi_{ik}(x) \qquad (m, n=1, 2, ...).$$

Let $\{\lambda_1(m): m=1, 2, ...\}$ and $\{\lambda_2(n): n=1, 2, ...\}$ be nondecreasing sequences of positive numbers, both tending to ∞. In this paper, we impose various conditions on the growth order of magnitude of these sequences. For the sake of convenience, here we list these properties (omitting the subscript):

$$\limsup_{m \to \infty} \frac{\lambda(2m)}{\lambda(m)} < \infty, \tag{1.3}$$

$$\limsup_{m \to \infty} \frac{\lambda(2m)}{\lambda(m)} < 2, \tag{1.4}$$

$$\limsup_{m \to \infty} \frac{\lambda(2m)}{\lambda(m)} < \sqrt{2}, \tag{1.5}$$

$$\liminf_{m \to \infty} \frac{\lambda(2m)}{\lambda(m)} > 1. \tag{1.6}$$

These conditions express that $\lambda(m)$ essentially behaves like an appropriate power of m multiplied by a slowly varying function. For instance, setting $\lambda(m) = m^{\alpha}[\log(m+1)]^{\beta}$ conditions (1.4) and (1.6) are satisfied if $0 < \alpha < 1$ and β is any real number.

The following two approximation theorems were proved in [4, Theorems 3 and 4].

THEOREM A. *If* $\{\lambda_1(m)\}$ *satisfies* (1.4), $\{\lambda_2(m)\}$ *satisfies* (1.3), *and*

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 \, [\log\log(i+3)]^2 \, [\log(k+1)]^2 \, [\lambda_1^2(i) + \lambda_2^2(k)] < \infty, \tag{1.7}$$

then

$$\sigma_{mn}^{10}(x) - f(x) = o_x\{\lambda_1^{-1}(m) + \lambda_2^{-1}(n)\} \qquad a.e. \tag{1.8}$$

THEOREM B. *If both* $\{\lambda_1(m)\}$ *and* $\{\lambda_2(n)\}$ *satisfy* (1.4), *and*

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 [\log\log(i+3)]^2 \, [\log\log(k+3)]^2 \, [\lambda_1^2(i) + \lambda_2^2(k)] < \infty, \tag{1.9}$$

then

$$\sigma_{mn}(x) - f(x) = o_x\{\lambda_1^{-1}(m) + \lambda_2^{-1}(n)\} \qquad a.e. \tag{1.10}$$

Unless it is specified otherwise, in the notation "o_x" we mean $\min\{m, n\} \to \infty$ (cf. [4, p. 109]). The logarithms are to the base 2.

The next theorem proved in [5] expresses a strong approximation property for double orthogonal series. The notion of strong approximation was introduced by Alexits [2] for single orthogonal series.

THEOREM C. *If both* $\{\lambda_1(m)\}$ *and* $\{\lambda_2(n)\}$ *satisfy* (1.5) *and condition* (1.9) *is also satisfied, then*

$$\left\{ m^{-1} n^{-1} \sum_{i=1}^{m} \sum_{k=1}^{n} [s_{ik}(x) - f(x)]^2 \right\}^{1/2} = o_x\{\lambda_1^{-1}(m) + \lambda_2^{-1}(n)\} \qquad a.e. \tag{1.11}$$

2. MAIN RESULTS: APPROXIMATION BY $\sigma_{mn}^{10}(x)$, $\sigma_{mn}(x)$ AND $s_{mn}(x)$

In the sequel, we assume that the sequences $\{\lambda_1(m)\}$ and $\{\lambda_2(n)\}$ satisfy one of the conditions (1.3)–(1.5). If, in addition, we assume the fulfillment of condition (1.6), then we can weaken conditions (1.7) and (1.9) and at the same time maintain conclusions (1.8), (1.10) and (1.11), respectively. This is the main concern of our paper.

In this manner, we get the improvements of Theorems A, B and C as follows.

THEOREM 1. *If* $\{\lambda_1(m)\}$ *satisfies* (1.4) *and* (1.6), $\{\lambda_2(n)\}$ *satisfies* (1.3) *and* (1.6), *and*

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2[\log(k+1)]^2 [\lambda_1^2(i)+\lambda_2^2(k)] < \infty, \tag{2.1}$$

then we have (1.8).

THEOREM 2. *If both* $\{\lambda_1(m)\}$ *and* $\{\lambda_2(n)\}$ *satisfy* (1.4) *and* (1.6), *and furthermore,*

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2[\log\log(i+3)]^2 [\lambda_1^2(i)+\lambda_2^2(k)] < \infty \tag{2.2}$$

and

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2[\log\log(k+3)]^2 [\lambda_1^2(i)+\lambda_2^2(k)] < \infty, \tag{2.3}$$

then we have (1.10).

THEOREM 3. *If both* $\{\lambda_1(m)\}$ *and* $\{\lambda_2(n)\}$ *satisfy* (1.5) *and* (1.6), *and conditions* (2.2)–(2.3) *are also satisfied, then we have* (1.11).

3. AUXILIARY RESULTS: APPROXIMATION BY $s_{2^p,n}(x)$ AND $s_{2^p,2^q}(x)$

The following two results will be used in the proofs of Theorems 1–3, but they are interesting in their own right, too.

THEOREM 4. *Under the conditions of Theorem* 1, *we have*

$$s_{2^p,n}(x)-f(x)=o_x\{\lambda_1^{-1}(2^p)+\lambda_2^{-1}(n)\} \qquad a.e. \tag{3.1}$$

THEOREM 5. *If* $\{\lambda_1(m)\}$ *satisfies* (1.4) *and* (1.6), $\{\lambda_2(n)\}$ *satisfies* (1.3) *and* (1.6), *and condition* (2.3) *is satisfied, then*

$$s_{2^p,2^q}(x)-f(x)=o_x\{\lambda_1^{-1}(2^p)+\lambda_2^{-1}(2^q)\} \qquad a.e. \tag{3.2}$$

Remark. If we interchange the conditions imposed on the sequences $\{\lambda_1(m)\}$ and $\{\lambda_2(n)\}$ in Theorem 5, then condition (2.2) (instead of (2.3)) implies the fulfillment of relation (3.2).

Before proving these two theorems, we emphasize the significance of condition (1.6). Namely, if a nondecreasing sequence $\{\lambda(m)\}$ of positive numbers satisfies (1.6), then

$$\sum_{p=0}^{m} \lambda^2(2^p) = \mathcal{O}\{\lambda^2(2^m)\} \qquad \text{for} \quad m = 0, 1, \ldots. \tag{3.3}$$

The proof of this inequality is a routine, therefore we omit it.

Proof of Theorem 4. *Part* 1. First we prove (3.1) in the special case where $n = 2^q$ with some integer $q \geqslant 0$. We make use of the decomposition

$$f(x) - s_{2^p, 2^q}(x) = \left\{ \sum_{i=2^p+1}^{\infty} \sum_{k=1}^{\infty} + \sum_{i=1}^{\infty} \sum_{k=2^q+1}^{\infty} - \sum_{i=2^p+1}^{\infty} \sum_{k=2^q+1}^{\infty} \right\} a_{ik} \phi_{ik}(x). \tag{3.4}$$

Accordingly, we divide Part 1 into three steps.

Step 1. If

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 \lambda_1^2(i) < \infty, \tag{3.5}$$

then

$$\sum_{i=2^p+1}^{\infty} \sum_{k=1}^{\infty} a_{ik} \phi_{ik}(x) = o_x\{\lambda_1^{-1}(2^p)\} \qquad \text{a.e. as } p \to \infty. \tag{3.6}$$

To see this, by (3.3) and (3.5),

$$\begin{aligned}
&\sum_{p=0}^{\infty} \lambda_1^2(2^p) \int \left[\sum_{i=2^p+1}^{\infty} \sum_{k=1}^{\infty} a_{ik} \phi_{ik}(x) \right]^2 d\mu(x) \\
&\quad = \sum_{p=0}^{\infty} \lambda_1^2(2^p) \sum_{i=2^p+1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 \\
&\quad = \sum_{i=2}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 \sum_{p:\, 2^p < i} \lambda_1^2(2^p) \\
&\quad = \mathcal{O}(1) \sum_{i=2}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2\, \lambda_1^2(i) < \infty.
\end{aligned}$$

Applying B. Levi's theorem gives (3.6).

Step 2. Similarly, if

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 \lambda_2^2(k) < \infty, \tag{3.7}$$

then

$$\sum_{i=1}^{\infty} \sum_{k=2^q+1}^{\infty} a_{ik}\phi_{ik}(x) = o_x\{\lambda_2^{-1}(2^q)\} \qquad \text{a.e. as } q \to \infty. \tag{3.8}$$

Step 3. If

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 \lambda_1^2(i) \log(k+1) < \infty, \tag{3.9}$$

then

$$\sum_{i=2^p+1}^{\infty} \sum_{k=2^q+1}^{\infty} a_{ik}\phi_{ik}(x) = o_x\{\lambda_1^{-1}(2^p)\} \qquad \text{a.e. as } \max\{p, q\} \to \infty. \tag{3.10}$$

To this end, by (3.3) and (3.9),

$$\begin{aligned}
\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \lambda_1^2(2^p) \int \left[\sum_{i=2^p+1}^{\infty} \sum_{k=2^q+1}^{\infty} a_{ik}\phi_{ik}(x) \right]^2 d\mu(x) \\
&= \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \lambda_1^2(2^p) \sum_{i=2^p+1}^{\infty} \sum_{k=2^q+1}^{\infty} a_{ik}^2 \\
&= \sum_{i=2}^{\infty} \sum_{k=2}^{\infty} a_{ik}^2 \sum_{p:\, 2^p < i} \lambda_1^2(2^p) \sum_{q:\, 2^q < k} 1 \\
&= \mathcal{O}(1) \sum_{i=2}^{\infty} \sum_{k=2}^{\infty} a_{ik}^2 \lambda_1^2(i) \log k < \infty.
\end{aligned}$$

Applying again B. Levi's theorem yields (3.10).

Collecting Steps 1–3, via (3.4), we find that if conditions (3.7) and (3.9) are satisfied, then we have (3.1) for $n = 2^q$:

$$s_{2^p, 2^q}(x) - f(x) = o_x\{\lambda_1^{-1}(2^p) + \lambda_2^{-1}(2^q)\} \qquad \text{a.e.} \tag{3.11}$$

Clearly, both (3.7) and (3.9) follow from (2.1).

Part 2. Let $2^q < n \leqslant 2^{q+1}$ with some $q \geqslant 0$. Then

$$s_{2^p, n}(x) - s_{2^p, 2^q}(x) = \left\{ \sum_{i=1}^{\infty} \sum_{k=2^q+1}^{n} - \sum_{i=2^p+1}^{\infty} \sum_{k=2^q+1}^{n} \right\} a_{ik}\phi_{ik}(x). \tag{3.12}$$

Accordingly, we divide Part 2 into two steps.

Step 4. If

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 \lambda_2^2(k)[\log(k+1)]^2 < \infty, \tag{3.13}$$

then

$$M_q(x) := \max_{2^q < n \leqslant 2^{q+1}} \left| \sum_{i=1}^{\infty} \sum_{k=2^q+1}^{n} a_{ik} \phi_{ik}(x) \right|$$
$$= o_x\{\lambda_2^{-1}(2^q)\} \qquad \text{a.e. as } q \to \infty. \tag{3.14}$$

Indeed, by the Rademacher–Menshov inequality (see, e.g., [1, p. 73] or [3, Theorem 3]),

$$\int [M_q(x)]^2 \, d\mu(x) \leqslant [\log 2^{q+1}]^2 \sum_{i=1}^{\infty} \sum_{k=2^q+1}^{2^{q+1}} a_{ik}^2.$$

Thus, by (3.13),

$$\sum_{q=0}^{\infty} \lambda_2^2(2^q) \int [M_q(x)]^2 \, d\mu(x)$$
$$= \mathcal{O}(1) \sum_{q=0}^{\infty} \lambda_2^2(2^q) \sum_{i=1}^{\infty} \sum_{k=2^q+1}^{2^{q+1}} a_{ik}^2 [\log 2k]^2$$
$$= \mathcal{O}(1) \sum_{i=1}^{\infty} \sum_{k=2}^{\infty} a_{ik}^2 \lambda_2^2(k) [\log 2k]^2 < \infty.$$

Hence, B. Levi's theorem implies (3.14).

Step 5. If

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 \lambda_1^2(i)[\log(k+1)]^2 < \infty, \tag{3.15}$$

then

$$M_{pq}(x) := \max_{2^q < n \leqslant 2^{q+1}} \left| \sum_{i=2^p+1}^{\infty} \sum_{k=2^q+1}^{n} a_{ik} \phi_{ik}(x) \right|$$
$$= o_x\{\lambda_1^{-1}(2^p)\} \qquad \text{a.e. as } p \to \infty, \tag{3.16}$$

uniformly in q.

In fact, again by the Rademacher–Menshov inequality,

$$\int [M_{pq}(x)]^2 \, d\mu(x) \leqslant [\log 2^{q+1}]^2 \sum_{i=2^p+1}^{\infty} \sum_{k=2^q+1}^{2^{q+1}} a_{ik}^2.$$

Thus, by (3.3) and (3.15),

$$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \lambda_1^2(2^p) \int [M_{pq}(x)]^2 \, d\mu(x)$$

$$= \mathcal{O}(1) \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \lambda_1^2(2^p) \sum_{i=2^p+1}^{\infty} \sum_{k=2^q+1}^{2^{q+1}} a_{ik}^2 [\log 2k]^2$$

$$= \mathcal{O}(1) \sum_{p=0}^{\infty} \lambda_1^2(2^p) \sum_{i=2^p+1}^{\infty} \sum_{k=2}^{\infty} a_{ik}^2 [\log 2k]^2$$

$$= \mathcal{O}(1) \sum_{i=2}^{\infty} \sum_{k=2}^{\infty} a_{ik}^2 [\log 2k]^2 \sum_{p:\, 2^p < i} \lambda_1^2(2^p)$$

$$= \mathcal{O}(1) \sum_{i=2}^{\infty} \sum_{k=2}^{\infty} a_{ik}^2 [\log 2k]^2 \, \lambda_1^2(i) < \infty.$$

Hence, we can conclude (3.16) via B. Levi's theorem.

Putting (3.12), (3.14), and (3.16) together yields the following: if (2.1) is satisfied, then

$$\max_{2^q < n \leqslant 2^{q+1}} |s_{2^p, n}(x) - s_{2^p, 2^q}(x)| = o_x\{\lambda_1^{-1}(2^p) + \lambda_2^{-1}(2^q)\} \qquad \text{a.e.} \tag{3.17}$$

Combining (3.11) and (3.17) results in (3.1) to be proved.

Proof of Theorem 5. It is an easy consequence of Theorem 4. To this effect, we introduce new coefficients and functions:

$$\tilde{a}_{ir} = \left\{ \sum_{k=2^{r-1}+1}^{2^r} a_{ik}^2 \right\}^{1/2} \qquad (r = 0, 1, ...)$$

and

$$\tilde{\phi}_{ir}(x) = \frac{1}{\tilde{a}_{ir}} \sum_{k=2^{r-1}+1}^{2^r} a_{ik} \phi_{ik}(x) \qquad \text{if} \quad \tilde{a}_{ir} \neq 0,$$

$$= \phi_{i, 2^r}(x) \qquad \text{if} \quad \tilde{a}_{ir} = 0.$$

It is not hard to check that $\{\tilde{\phi}_{ir}(x): i = 1, 2, ...; r = 0, 1, ...\}$ is also an ONS, and by (2.3),

$$\sum_{i=1}^{\infty} \sum_{r=0}^{\infty} \tilde{a}_{ir}^2 [\log(r+2)]^2 \, [\lambda_1^2(i) + \lambda_2^2(2^r)]$$

$$= \mathcal{O}(1) \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 [\log \log 8k]^2 \, [\lambda_1^2(i) + \lambda_2^2(k)] < \infty.$$

This means that condition (2.1) in Theorem 1 is satisfied. Thus, by (3.1),

$$\tilde{s}_{2^p, q}(x) - f(x) = o_x\{\lambda_1^{-1}(2^p) + \lambda_2^{-1}(2^q)\} \qquad \text{a.e.}, \tag{3.18}$$

where

$$\begin{aligned}\tilde{s}_{2^p, q}(x) &= \sum_{i=1}^{2^p} \sum_{r=0}^{q} \tilde{a}_{ir} \tilde{\phi}_{ir}(x)\\ &= \sum_{i=1}^{2^p} \sum_{k=1}^{2^q} a_{ik} \phi_{ik}(x) = s_{2^p, 2^q}(x),\end{aligned}$$

i.e., (3.18) is equivalent to (3.2) to be proved.

4. Proofs of Theorems 1–3

We will repeat certain parts of the proofs of the corresponding theorems in [4, 5], respectively, while making use of Theorems 4 and 5 in the present paper to estimate the terms involving $s_{2^p, n}(x)$ and $s_{2^p, 2^q}(x)$.

Proof of Theorem 1. Let $2^p \leqslant m < 2^{p+1}$ with an integer $p \geqslant 0$. Then we can write

$$\begin{aligned}\sigma_{mn}^{10}(x) - f(x) = {} & [\sigma_{mn}^{10}(x) - \sigma_{2^p, n}^{10}(x)]\\ & + [\sigma_{2^p, n}^{10}(x) - s_{2^p, n}(x)]\\ & + [s_{2^p, n}(x) - f(x)].\end{aligned} \tag{4.1}$$

The third term on the right is estimated in Theorem 4. We note that condition (1.6) is actually used only in this estimate.

As for the first and second terms on the right in (4.1), we refer to [4, p. 121 and p. 123], respectively:

(i) Under condition (3.15) (which is weaker than (2.1)), we have

$$\sigma_{2^p, n}^{10}(x) - s_{2^p, n}(x) = o_x\{\lambda_1^{-1}(2^p)\} \qquad \text{a.e. as } p \to \infty \tag{4.2}$$

and

$$\max_{2^p < m \leqslant 2^{p+1}} |\sigma_{mn}^{10}(x) - \sigma_{2^p, n}^{10}(x)| = o_x\{\lambda_1^{-1}(2^p)\} \qquad \text{a.e. as } p \to \infty, \tag{4.3}$$

uniformly in n, in both cases.

Now, (4.1), (3.1), (4.2) and (4.3) imply (1.8).

Proof of Theorem 2. Let $2^p \leqslant m \leqslant 2^{p+1}$ and $2^q \leqslant n \leqslant 2^{q+1}$ with integers $p \geqslant 0$ and $q \geqslant 0$. We consider the following representation:

$$\begin{aligned}
&\sigma_{mn}(x) - f(x) \\
&\quad = [\sigma_{mn}(x) - \sigma_{m,2^q}(x) - \sigma_{2^p,n}(x) + \sigma_{2^p,2^q}(x)] \\
&\qquad + [\sigma_{m,2^q}(x) - \sigma_{2^p,2^q}(x)] + [\sigma_{2^p,n}(x) - \sigma_{2^p,2^q}(x)] \\
&\qquad + [\sigma_{2^p,2^q}(x) - s_{2^p,2^q}(x)] + [s_{2^p,2^q}(x) - f(x)].
\end{aligned} \tag{4.4}$$

We have estimated the fifth term on the right in Theorem 5. As for the other terms, we refer again to [4]:

(ii) (see [4, pp. 126–127]): If the conditions

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 [\log \log(i+3)]^2 \lambda_2^2(k) < \infty \tag{4.5}$$

and

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 [\log \log(k+3)]^2 \lambda_1^2(i) < \infty \tag{4.6}$$

are satisfied, then

$$\sigma_{2^p,2^q}(x) - s_{2^p,2^q}(x) = o_x\{\lambda_1^{-1}(2^p) + \lambda_2^{-1}(2^q)\} \qquad \text{a.e.}$$

(iii) (see [4, p. 128]): Under condition (4.6), we have

$$\max_{2^p < m \leqslant 2^{p+1}} |\sigma_{m,2^q}(x) - \sigma_{2^p,2^q}(x)| = o_x\{\lambda_1^{-1}(2^p)\} \qquad \text{a.e. as } p \to \infty,$$

uniformly in q.

(iv) (see [4, p. 129]): Under condition (4.5), we have

$$\max_{2^q < n \leqslant 2^{q+1}} |\sigma_{2^p,n}(x) - \sigma_{2^p,2^q}(x)| = o_x\{\lambda_2^{-1}(2^q)\} \qquad \text{a.e. as } q \to \infty,$$

uniformly in p.

(v) (see [4, p. 130]): If conditions (3.5) and (3.7) are satisfied, i.e., if

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 [\lambda_1^2(i) + \lambda_2^2(k)] < \infty,$$

then

$$\begin{aligned}
&\max_{2^p \leqslant m \leqslant 2^{p+1}} \max_{2^q \leqslant n \leqslant 2^{q+1}} |\sigma_{mn}(x) - \sigma_{m,2^q}(x) - \sigma_{2^p,n}(x) + \sigma_{2^p,2^q}(x)| \\
&\quad = o_x\{\min\{\lambda_1^{-1}(2^p), \lambda_2^{-1}(2^q)\}\} \qquad \text{a.e. as } \max\{p, q\} \to \infty.
\end{aligned}$$

Estimates (ii)–(v) and (3.2), via (4.4), clearly imply Theorem 2.

Proof of Theorem 3. By the triangle inequality,

$$
\begin{aligned}
&\left\{m^{-1}n^{-1} \sum_{i=1}^{m} \sum_{k=1}^{n} [s_{ik}(x) - f(x)]^2\right\}^{1/2} \\
&\quad \leqslant \left\{m^{-1}n^{-1} \sum_{i=1}^{m} \sum_{k=1}^{n} [s_{ik}(x) - \sigma_{ik}(x)]^2\right\}^{1/2} \\
&\qquad + \left\{m^{-1}n^{-1} \sum_{i=1}^{m} \sum_{k=1}^{n} [\sigma_{ik}(x) - f(x)]^2\right\}^{1/2} \\
&\quad =: \delta_{mn}^{(1)}(x) + \delta_{mn}^{(2)}(x).
\end{aligned}
$$

Accordingly, we accomplish the proof in two parts.

Part 1 (see [5, Theorem 2]). If conditions (1.5), (4.5) and (4.6) are satisfied, then

$$\delta_{mn}^{(1)}(x) = o_x\{\lambda_1^{-1}(m) + \lambda_2^{-1}(n)\} \qquad \text{a.e.}$$

Part 2. The conditions in Theorem 3 clearly imply the fulfillment of those in Theorem 2. Thus, we have (1.10). Then by [5, Lemma 1],

$$\delta_{mn}^{(2)}(x) = o_x\{\lambda_1^{-1}(m) + \lambda_2^{-1}(n)\} \qquad \text{a.e.}$$

The proof of Theorem 3 is complete.

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